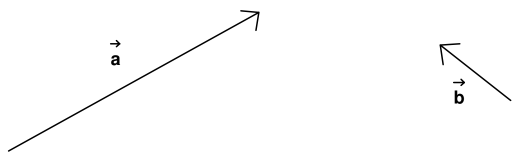
VECTORS-IB -SL

- **A vector quantity has direction and magnitude.**
  Examples of vector quantities : Force, velocity, acceleration etc.

- **A vector is a directed line segment used to represent a vector quantity.**

  Examples:

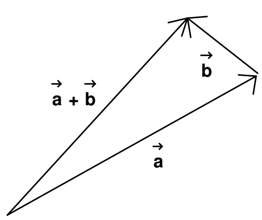


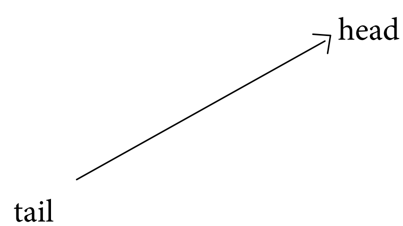
- Vectors do not have a fixed position in the coordinate plane, so they can be translated (moved) without changing their meaning.

Adding vectors

- To add vectors geometrically, translate the second vector so that it starts at the end of the first vector. The vector sum goes from the head of the first vector to the tailof the second.

  Example:





- Vectors can be represented numerically. The numeric form of a vector in two ways:

column form

$$\vec{a} = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix}$$

component form.

$$\vec{b} = 4\vec{i} - 5\vec{j} + 2\vec{k}$$

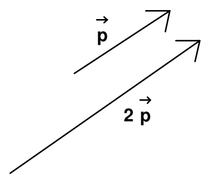
- To add the vectors, simply add their corresponding components.

  Example: Add vectors $\vec{v} = 3\vec{i} - 4\vec{j}$ and $\vec{w} = 5\vec{i} + 2\vec{j}$.

$$\vec{v} + \vec{w} = 8\vec{i} - 2\vec{j}$$

- A <u>scalar</u> is a number. It has magnitude but no direction. Vectors can be multiplied by scalars.

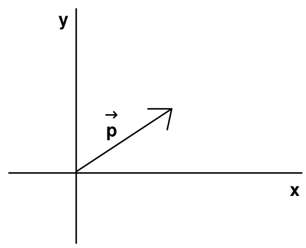
Examples:



$$\vec{m} = -6\vec{i} + 2\vec{j} + 5\vec{k}$$
$$\tfrac{1}{2}\vec{m} = -3\vec{i} + \vec{j} + \tfrac{5}{2}\vec{k}$$

- A <u>position</u> <u>vector</u> is a vector that starts at the origin (0, 0).

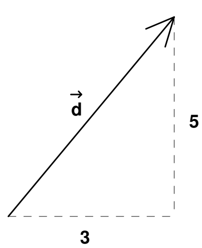
Example:



Vector $\vec{p}$ drawn as a position vector.

- The <u>magnitude</u> (length) of a vector can be calculated using the Pythagoras Theorem.

Example:



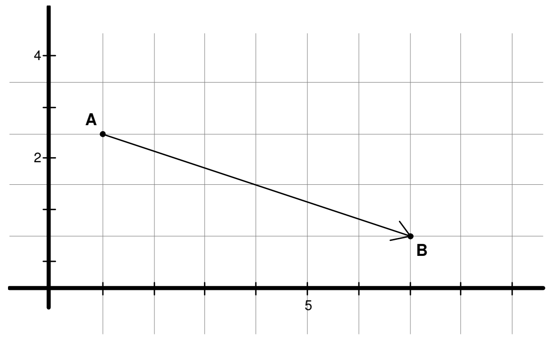
$$\vec{d} = 3\vec{i} + 5\vec{j}$$
$$\left|\vec{d}\right| = \sqrt{3^2 + 5^2}$$
$$= \sqrt{9 + 25}$$
$$= \sqrt{34}$$

Example: Find the length of $\vec{c} = -2\vec{i} - 4\vec{j}$.

$$|\vec{c}| = \sqrt{(-2)^2 + (-4)^2}$$
$$= \sqrt{4 + 16}$$
$$= \sqrt{20}$$

- **A displacement vector is a vector between two points.**
- To find a displacement vector numerically, subtract the coordinates: "head point minus stail coordinates"

Example:1 Draw the displacement vector from $A(1, 3)$ to $B(7, 1)$. Then write vector $\overrightarrow{AB}$ in component form.



Vector $\overrightarrow{AB} = 6\vec{i} - 2\,j$

Example:2 Find the displacement vector from $C(-3, 5, -1)$ to $D(4, -2, 7)$.

$$\overrightarrow{CD} = (4-(-3))\vec{i} + (-2-5)\vec{j} + (7-(-1))\vec{k}$$
$$= 7\vec{i} - 7\vec{j} + 8\vec{k}$$

Unit Vectors

- **A unit vector is a vector that is one unit long.**

Example:1 Determine whether or not each of the following vectors is a unit vector:

a) $\vec{v} = \vec{i} + \vec{j}$ b) $\vec{w} = \frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$ c) $\vec{m} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$

$|\vec{v}| = \sqrt{1^2 + 1^2} = \sqrt{2} \approx 1.4$ $\vec{v}$ is not a unit vector.

$|\vec{w}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}} \approx 0.7$ $\vec{w}$ is not a unit vector.

$|\vec{m}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{1} = 1$ $\vec{m}$ is a unit vector.

- **To find a unit vector parallel to a given vector, divide the vector by its length.**

Example:1 Find a unit vector parallel to $\vec{a} = 3\vec{i} + 8\vec{j}$

$$|\vec{a}| = \sqrt{3^2 + 8^2} = \sqrt{73}$$
$$\vec{u} = \frac{\vec{a}}{|\vec{a}|} = \frac{3}{\sqrt{73}}\vec{i} + \frac{8}{\sqrt{73}}\vec{j}$$

The Dot Product /Scalar product

- **One way of multiplying two vectors is to find the <u>dot product</u>. The dot** product of vectors $\vec{a}$ and $\vec{b}$ is written $\vec{a} \cdot \vec{b}$.
- **The dot product is also called the <u>scalar product</u> because the result is a scalar, not a vector.**

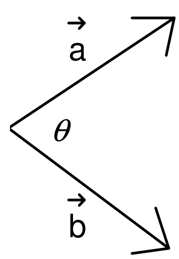
- To find the dot product of two vectors, multiply the corresponding components and add them up.

  Example: Find the dot product of $\vec{a} = 3\vec{i} - 2\vec{j} - 4\vec{k}$ and $\vec{b} = 2\vec{i} + \vec{j} - \vec{k}$.

$$\begin{aligned} \vec{a} \cdot \vec{b} &= 3(2) + (-2)(1) + (-4)(-1) \\ &= 6 - 2 + 4 = 8 \end{aligned}$$

Application of dot product of vectors

- **The dot product is used to find the angle between two vectors when they are placed tail-to-tail. The formula is:**



$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

Example: Find the angle (in degrees) between $\vec{c} = \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$ and $\vec{d} = \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix}$.

$$\vec{c} \cdot \vec{d} = 3(-2) + 1(5) + 7(1) = 6$$

$$|\vec{c}| = \sqrt{3^2 + 1^2 + 7^2} = \sqrt{59}$$

$$|\vec{d}| = \sqrt{(-2)^2 + 5^2 + 1^2} = \sqrt{30}$$

$$\cos\theta = \frac{6}{\sqrt{59} \cdot \sqrt{30}} \approx 0.143$$

$$\theta = \cos^{-1}(0.143) \approx 81.801^\circ$$

Perpendicular vectors

- **Vectors are perpendicular when the angle between them is a right angle.**
  Numerically, vectors are perpendicular if and only if their dot product equals zero (since $\cos 90^\circ = 0$).

Example:1 Show that vectors $\vec{g} = 2\vec{i} - 15\vec{j} - \vec{k}$ and $\vec{h} = 6\vec{i} + \vec{j} - 3\vec{k}$ are perpendicular.

$\vec{g} \cdot \vec{h} = 2(6) + (-15)(1) + (-1)(-3) = 0$ Hence the vectors are perpendicular

Example:2 Find the value of k so that vectors $\vec{a} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ -10 \\ k \end{pmatrix}$ are perpendicular.

$$\vec{a} \cdot \vec{b} = 1(1) + (1)(-10) + (-3)(k) = -9 - 3k$$

$$-9 - 3k = 0$$

$$-3k = 9$$

$$k = -3$$

Parallel vectors

- **Vectors are parallel when they have the same direction (or opposite directions).**
  Numerically,
  Vectors are parallel if and only if they are scalar multiples of each other.

Example: Find the value of k so that vectors $\vec{c} = \begin{pmatrix} -12 \\ -8 \\ k \end{pmatrix}$ and $\vec{d} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ are parallel.

$$-12 = 3(-4) \text{ and } -8 = 2(-4) \text{ so } k = 1(-4) = -4$$

Vector Equations of Lines

- You already know how to write the Cartesian equation, $y = mx + b$, of a line if you know the slope and any point on the line.

- Lines can also be defined by vector equations. To write the vector equation of a line you need to know a point on the line and the direction vector for the line (a vector parallel to the line).

- The general form of the vector equation of a line is

$$\vec{r} = \vec{a} + t\vec{b}$$

where $\vec{a}$ is the position vector to a point on the line, $\vec{b}$ is the direction vector for the line, t is an arbitrary scalar (the independent variable) and $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (the dependent variable).

Example: Write a vector equation of the line passing through (2, –1) and parallel to $3\vec{i} - 5\vec{j}$.

$$\vec{r} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

Example: Write a vector equation of the line passing through (–1, 3, 7) and parallel to $5\vec{i} - 2\vec{j} + \vec{k}$.

$$\vec{r} = \begin{pmatrix} -1 \\ 3 \\ 7 \end{pmatrix} + t\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

Example: Write a vector equation of the line passing through the points A(2, –1, 5) and B(3, 1, 2)

a direction vector is $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

so an equation is $\vec{r} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

Application of vectors equation Vectors

Application -1

- When you know the vector equation of a line, you can find points on the line by choosing any number for t and plugging it in to the equation.

Example: Write down three points on the line $\vec{r} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

when $t = 1$, the point is (4, 4, 3)

when $t = -1$, the point is (–2, 6, 1)

when $t = 2$, the point is (7, 3, 4)

(Remember these are just a few examples – you can pick any number you want for t!)

Application 2

- If two vector lines intersect, you can find the point of intersection by setting the equations equal to each other, solving for one of the independent variables, and then plugging back in to the equation to get the coordinates of the point.

Example: The lines $\vec{r}_1 = \begin{pmatrix} -2 \\ -3 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\vec{r}_2 = \begin{pmatrix} 6 \\ 5 \end{pmatrix} + s\begin{pmatrix} -2 \\ -4 \end{pmatrix}$ intersect at the point P. Find the coordinates of P.

$$\begin{aligned} -2+3t &= 6-2s \\ -3-2t &= 5-4s \end{aligned} \quad \rightarrow \quad \begin{aligned} -4+6t &= 12-4s \\ \underline{-(-3-2t} &\underline{= 5-4s)} \end{aligned}$$

Solving any two equations simultaneously we get:

$$\begin{aligned} -4+6t &= 12-4s \\ -3-2t &= 5-4s \end{aligned}$$

$$\begin{aligned} -1+8t &= 7 \\ 8t &= 8 \\ t &= 1 \end{aligned}$$

Plugging in the values we get

$$\begin{aligned} x &= -2+1(3) = 1 \\ y &= -3+1(-2) = -5 \end{aligned}$$

The coordinates of $\underline{P}$ are (1, –5)